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On the Jumping Constant Conjecture for Multigraphs

VOJTĚCH RÖDL

Department of Mathematics and Computer Science, Emory University, Atlanta

AND

ALEXANDER SIDORENKO*

Courant Institute of Mathematical Sciences, New York University, New York

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Let $\mathscr{G}$ be an infinite family of graphs closed under taking subgraphs. For each n, set

$$\varepsilon(n, \mathscr{G}) = \max\{|E(G)|: G \in \mathscr{G}, |V(G)| = n\}.$$

Let density $\tau(\mathscr{G})$ of a family $\mathscr{G}$ be defined by

$$\tau(\mathscr{G}) = \lim_{n \to \infty} \frac{\varepsilon(n, \mathscr{G})}{\binom{n}{2}}.$$

The set of all possible densities of graph families was determined by Erdös and Stone in 1946. The analogous problem for multigraphs with multiplicity of edges not exceeding q (where $q = 2, 3, \dots$) appears to be much harder. In 1973 Brown, Erdös, and Simonovits conjectured that the set of all possible densities is well-ordered. They verified their conjecture for $q = 2$. If $q = 1$, this follows from the Erdös–Stone theorem. We disprove this conjecture for all $q \geq 4$. © 1995 Academic Press, Inc.

1. THE JUMPING CONSTANT CONJECTURE AND TURÁN PROBLEM FOR MULTIGRAPHS

We fix a positive integer q and consider finite multigraphs (without loops) in which any two vertices are joined by at most q edges. The vertex-set of a multigraph G is denoted by $V(G)$. For a simple graph G (where multiple edges are not allowed), we denote the set of edges by $E(G)$.

* E-mail address: sidorenk@cims.nyu.edu.

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Let $\mathscr{F}$ be a (finite or infinite) family of multigraphs. We assume that every $F \in \mathscr{F}$ has at least one edge. Let Forb($\mathscr{F}$) denote the class of q-multigraphs which contain no $F \in \mathscr{F}$ as a subgraph (not necessarily induced). Denote by ex$(n, \mathscr{F})$ the maximum number of edges (counting their multiplicities) in a multigraph on n vertices from the class Forb($\mathscr{F}$). Since the ratio ex$(n, \mathscr{F})/\binom{n}{2}$ does not increase when n increases, there exists the limit

$$\tau(\mathscr{F}) = \lim_{n\to\infty} \frac{\operatorname{ex}(n, \mathscr{F})}{\binom{n}{2}}.$$

The Turán problem is to determine ex$(n, \mathscr{F})$ for every n. This problem is very hard, thus its asymptotic version, to determine $\tau(\mathscr{F})$, is usually considered.

If $q = 1$ (simple graphs) then $\tau(\mathscr{F})$ can be described in terms of chromatic numbers of graphs $F \in \mathscr{F}$. If $\chi = \chi(\mathscr{F}) = \min\{\chi(F): F \in \mathscr{F}\}$ then Forb($\mathscr{F}$) clearly contains arbitrarily large complete $(\chi - 1)$-partite graphs whose density is $1 - 1/(\chi - 1) - o(1)$ as the size of the color classes tends to infinity. Hence $\tau(\mathscr{F}) \geq 1 - 1/(\chi - 1)$. It was proven in [6] that the opposite inequality holds as well and thus $\tau(\mathscr{F}) = 1 - 1/(\chi - 1)$ (cf. [5]). This implies that the set of all possible values τ is $\{1 - 1/n, n = 1, 2, \dots\} \cup \{1\}$. For $q \geq 2$, determining $\tau(\mathscr{F})$ is much more complicated.

A multigraph G can be represented by its adjacency matrix $A = [a_{ij}]$, where a_{ij} equals the number of edges which join v_i and v_j. We define

$$\lambda(G) = \max\left\{ \sum_{ij} a_{ij} x_i x_j \,\middle|\, x_1 + x_2 + \dots = 1, x_i \geq 0 \right\}. \tag{1}$$

One may show that $\lambda(G)$ is always a rational number.

The function λ was introduced in [8] for graphs, and was extended for multigraphs and hypergraphs in [1–3, 7, 10, 11]. This function is a convenient tool when we want to investigate the set T_q of all possible values $\tau(\mathscr{F})$, where $\mathscr{F}$ is a family of q-multigraphs.

Let L_q be the set of all possible values of $\lambda(G)$ where G is a q-multigraph, and L_q^{up} be the set of limits of all non-decreasing sequences with elements in L_q. The following theorem gives a description of the set T_q in terms of $\lambda(G)$:

THEOREM 1.1 ([2, Theorem 4]). *For every* $q \geq 1$

$$T_q = L_q^{up}.$$

It was conjectured in [1–3] that the set L_q is well-ordered (under the usual ordering of reals) for every $q > 1$. This is known as the *jumping constant conjecture*. Namely, a real α is a *jump* if there exists $\beta > \alpha$ such that $\lambda(G) > \alpha$ implies $\lambda(G) \geq \beta$. So the well-ordering means that every real number is a jump. This conjecture was verified for $q = 2$ in [3, 11]. In this case, finite algorithmic procedures were found for determining $\tau(\mathscr{F})$ for any given $\mathscr{F}$.

In the present work, we disprove the conjecture by showing that for q-multigraphs with $q \geq 4$, there are infinitely many $\alpha \in [3, q]$ which are not jumps. The analogous conjecture for hypergraphs was disproven earlier in [7]. Another related work is [11], where it was shown that for every $q \geq 3$, there are infinitely many q-multigraphs G having the same value of λ: $\lambda(G) = q - 1$. Our disproof of the jumping constant conjecture for multigraphs combines the main ideas of both papers. The crucial step to complete the disproof is provided by Theorem 2.2, where a distance representation of trees in an Euclidean space is constructed.

It was also conjectured in [1–3] that any infinite family $\mathscr{F}$ of q-multigraphs contains a finite subfamily $\mathscr{F}'$ such that $\tau(\mathscr{F}) = \tau(\mathscr{F}')$. This can be expressed in terms of L_q as follows:

THEOREM 1.2 ([2, Theorem 5]). *The following three statements are equivalent*:

(i) *The set L_q of reals which can be realized as $\lambda(G)$ is well-ordered*;

(ii) *The set T_q of reals which can be realized as $\tau(\mathscr{F})$ is well-ordered*;

(iii) (*Compactness property*) *Any infinite family $\mathscr{F}$ of q-multigraphs contains a finite subfamily $\mathscr{F}'$ such that $\tau(\mathscr{F}) = \tau(\mathscr{F}')$.*

For $q \geq 4$, the statement (i) will be disproved in the next section. Thus the compactness property does not hold and the set T_q also is not well-ordered.

By the "Transfer Principle" of [2, Lemma 4], our results imply that statements (i)–(iii) also do not hold for oriented q-multigraphs with $q \geq 2$.

2. MAIN RESULTS

In this section, we list our main results. Proofs appear in Section 3.

As noted in [11], $\lambda(G)$ can be interpreted geometrically on the basis of Schoenberg's theorem [9]. We use this interpretation as one of our main tools (for more details, see [11]).

THEOREM 2.1. *Let points $\mathbf{z}_1, z_2, \ldots, \mathbf{z}_n$ of a Euclidean space span a d-dimensional hyperplane and lay on a $(d - 1)$-dimensional sphere of radius*

ρ. Set $a_{ij} = \|\mathbf{z}_i - \mathbf{z}_j\|^2$. Then

$$\max\left\{ \sum_{i,j=1}^{n} a_{ij}x_ix_j \middle| x_1 + x_2 + \cdots + x_n = 1 \right\} = 2\rho^2.$$

We call a simple graph a *p-tree* if one is a tree where at most one vertex has degree $p + 1$ and the degrees of all others are at most p.

THEOREM 2.2. *Let $p \geq 3$. For any p-tree T with $V(T) = \{v_1, v_2, \ldots, v_n\}$, there exist points $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n$ in a Euclidean space such that*

$$\|\mathbf{z}_i - \mathbf{z}_j\|^2 = \begin{cases} p + 1 & \text{if } \{v_i, v_j\} \in E(T), \\ p & \text{if } \{v_i, v_j\} \notin E(T), \end{cases}$$

and the square radius of the sphere circumscribed to $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n$ is strictly smaller than $\frac{1}{2}p$.

If G is a simple graph, denote by G^p a multigraph on the same vertices such that any two vertices are joined in G^p by $p + 1$ or p edges depending on whether they are joined in G.

Consider vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n$ from Theorem 2.2. Then by Theorem 2.1,

$$\lambda(T^p) \leq \max\left\{ \sum_{i,j=1}^{n} \|\mathbf{z}_i - \mathbf{z}_j\|^2 x_ix_j \middle| x_1 + x_2 + \cdots + x_n = 1 \right\} \leq 2\rho^2,$$

where $\rho^2 < \frac{1}{2}p$ according to Theorem 2.2. Hence we have the following

COROLLARY 2.3. *If $3 \leq p < q$, then for any p-tree T, the inequality $\lambda(T^p) < p$ holds.*

We note that the requirement $p \geq 3$ is essential. In particular, let T_m be a combination of 3 paths of length m (i.e., with m edges each) having a common endpoint. This is a 2-tree. However, $\lambda(T_3^2) = 2\frac{1}{106}$.

It was shown in [11] that the equality $\lambda(G^p) = p$ holds for any connected regular graph G of degree p. We are going to construct infinite series of graphs G_n with the property $\lambda(G_n^p) > p$, $\liminf_{n\to\infty} \lambda(G_n^p) = p$ by choosing G_n to be a slight modification of a regular graph without short cycles.

THEOREM 2.4. *For every k and p, there exists a graph G where two vertices have degree $p + 1$, the distance between them is at least k, all other vertices have degree p, and there are no cycles of length k or less.*

For multigraphs $G_1, G_2, \ldots, G_m$, let $\mathrm{Forb}(G_1, G_2, \ldots, G_m)$ denote the class of multigraphs which do not contain $G_1, G_2, \ldots, G_m$ as subgraphs.

THEOREM 2.5. *A real α is a jump if and only if there exists a finite family of multigraphs $G_1, G_2, \ldots, G_m$ such that*

$$\mathrm{Forb}(G_1, G_2, \ldots, G_m) = \{G : \lambda(G) \leq \alpha\}.$$

LEMMA 2.6. *Let $t \geq q$, and $G_0, G_1, \ldots, G_l$ be disjoint multigraphs such that $\lambda(G_i) = t - 1/\varepsilon_i$, $\varepsilon_i > 0$ $(i = 1, 1, \ldots, l)$. Let a multigraph G be formed from $G_0, G_1, \ldots, G_l$ by adding t multiple edges $\{x, y\}$ for every $x \in V(G_i)$, $y \in V(G_j)$ such that $i \neq j$. Then $\lambda(G) = t - 1/\varepsilon$ where $\varepsilon = \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_l$.*

Now we are ready to prove the main result.

THEOREM 2.7. *None of the integers $3, 4, \ldots, q-1$ is a jump for q-multigraphs with $q \geq 4$.*

Proof. Suppose an integer $p \in \{3, 4, \ldots, q-1\}$ is a jump. Set $k = \max\{|V(G_i)| : i = 1, 2, \ldots m\}$, where $G_1, G_2, \ldots G_m$ are the multigraphs from the statement of Theorem 2.5 with $\alpha = p$. For this value k, there exists a graph G satisfying Theorem 2.4. Choosing all variables x_i in (1) to be equal to $1/|V(G)|$, we have

$$\lambda(G^p) \geq 2\left(p\binom{|V(G)|}{2} + |E(G)|\right)\frac{1}{|V(G)|^2} = p + \frac{2}{|V(G)|^2} > p.$$

To get a contradiction, it is enough to show that every induced subgraph H of G with k or less vertices satisfies $\lambda(H^p) \leq p$. As G does not contain cycles of length $|V(H)|$ or less, the graph H is a forest. Since the distance between the two vertices of degree $p+1$ in G is at least $|V(H)|$, they cannot be in the same component of the forest. Thus all the components $H_0, H_1, \ldots, H_l$ are p-trees. By Corollary 2.3, $\lambda(H_i^p) < p$ for every $i = 0, 1, \ldots, l$; then by Lemma 2.6, $\lambda(H^p) < p$. ∎

Moreover, we may find a lot of other reals which are not jumps also. A real α is not a jump iff there exist infinite series of multigraphs $G^{(m)}$ such that $\lambda(G^{(1)}) > \lambda(G^{(2)}) > \ldots$ and $\liminf_{m \to \infty} \lambda(G^{(m)}) = \alpha$. Substituting $G^{(m)}$ instead of G_0 in the statement of Lemma 2.6, we derive

COROLLARY 2.8. *Let t be an integer, $t \leq q$, and $\lambda(G_i) = t - 1/\varepsilon_i$, $\varepsilon_i > 0$ for $i = 1, 2, \ldots, l$. If $t - 1/\varepsilon_0$ is not a jump then $t - 1/(\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_l)$ is not a jump either.*

For example, let each of $G_1, G_2, \ldots, G_l$ consist of one vertex, so $\lambda(G_1) = \lambda(G_2) = \cdots = \lambda(G_l) = 0$. Then Theorem 2.7 and Corollary 2.8 imply that $t\,(1 - 1/(l + t/(t-p)))$ is not a jump for q-multigraphs with any l and any $3 \leq p < t \leq q$.

We note that all discovered non-jump values α satisfy $\alpha \geq 3$.

Theorem 1.2 and Lemma 2.6 also yield

COROLLARY 2.9. *Compactness property does not hold for q-multigraphs with $q \geq 4$.*

3. PROOFS OF THEOREMS

Proof of Theorem 2.1. Let $\mathbf{z}$ be the center of the sphere. Since $\mathbf{z}$ belongs to the hyperplane spanned by $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n$, there exist weights $x_1, x_2, \ldots, x_n$ such that $\sum_{i=1}^n x_i = 1$ and $\sum_{i=1}^n x_i \mathbf{z}_i = \mathbf{z}$. Set $\mathbf{y}_i = \mathbf{z}_i - \mathbf{z}$. Then $\sum_{i=1}^n x_i \mathbf{y}_i = 0$. For any $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
\sum_{j=1}^n x_j \|\mathbf{z}_i - \mathbf{z}_j\|^2 &= \sum_{j=1}^n x_j \|\mathbf{y}_i - \mathbf{y}_j\|^2 = \sum_{j=1}^n x_j \big(\|\mathbf{y}_i\|^2 + \|\mathbf{y}_j\|^2 - 2(\mathbf{y}_i, \mathbf{y}_j)\big) \\
&= \sum_{j=1}^n x_j \big(\rho^2 + \rho^2 - 2(\mathbf{y}_i, \mathbf{y}_j)\big) \\
&= 2\rho^2 - 2\left(\mathbf{y}_i, \sum_{j=1}^n x_j \mathbf{y}_j\right) = 2\rho^2.
\end{aligned}$$

The value of the quadratic form at the point $(x_1, x_2, \ldots, x_n)$ is

$$\sum_{i,j=1}^n \|\mathbf{z}_i - \mathbf{z}_j\|^2 x_i x_j = \sum_{i=1}^n \left(\sum_{j=1}^n \|\mathbf{z}_i - \mathbf{z}_j\|^2 x_j \right) x_i = \sum_{i=1}^n 2\rho^2 x_i = 2\rho^2.$$

Now we show that at any other point $(x_1', x_2', \ldots, x_n') = (x_1 + \varepsilon_1, x_2 + \varepsilon_2, \ldots, x_n + \varepsilon_n)$ with $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n = 0$, the value of the form does not exceed $2\rho^2$:

$$\begin{aligned}
&\sum_{i,j=1}^{n} \|\mathbf{z}_i - \mathbf{z}_j\|^2 (x_i + \varepsilon_i)(x_j + \varepsilon_j) \\
&\quad = \sum_{i,j=1}^{n} \|\mathbf{z}_i - \mathbf{z}_j\|^2 x_i x_j + \sum_{i=1}^{n} \varepsilon_i \left(\sum_{j=1}^{n} \|\mathbf{z}_i - \mathbf{z}_j\|^2 x_j \right) \\
&\qquad + \sum_{j=1}^{n} \varepsilon_j \left(\sum_{i=1}^{n} \|\mathbf{z}_i - \mathbf{z}_j\|^2 x_i \right) + \sum_{i,j=1}^{n} \|\mathbf{z}_i - \mathbf{z}_j\|^2 \varepsilon_i \varepsilon_j \\
&\quad = 2\rho^2 + \sum_{i=1}^{n} \varepsilon_i 2\rho^2 + \sum_{j=1}^{n} \varepsilon_j 2\rho^2 + \sum_{i=1}^{n} \varepsilon_i \|\mathbf{z}_i\|^2 \sum_{j=1}^{n} \varepsilon_j \\
&\qquad + \sum_{j=1}^{n} \varepsilon_j \|\mathbf{z}_j\|^2 \sum_{i=1}^{n} \varepsilon_i - 2 \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j (\mathbf{z}_i, \mathbf{z}_j) \\
&\quad = 2\rho^2 - 2 \left\| \sum_{i=1}^{n} \varepsilon_i \mathbf{z}_i \right\|^2 \le 2\rho^2. \quad \blacksquare
\end{aligned}$$

Proof of Theorem 2.2. Let us extend the tree T to a p-tree T^* by adding new leaves such that (i) one vertex of T has degree $p + 1$ in T^* and (ii) all other vertices of T have degree in T^* equal to p. We choose the only vertex of degree $p + 1$ in T^* as a root, and orient all edges of T^* in the direction from the root. We enumerate all vertices $v_1, v_2, \ldots, v_m$ of T^* so that every oriented edge (v_i, v_j) satisfies $i < j$ and all vertices of $V(T^*) \setminus V(T)$ are in the end of the sequence. So v_1 is the root, and $V(T) = \{v_1, v_2, \ldots, v_n\}$, $n \le m$. We define vectors $\mathbf{z}_i = (z_{i1}, z_{i2}, \ldots, z_{im})$ with $i = 1, 2, \ldots, n$ by

$$z_{ij} = \begin{cases} -a & \text{if } i = j = 1, \\ -b & \text{if } i = j > 1, \\ c & \text{if } (i, j) \text{ is an oriented edge}, \\ 0 & \text{otherwise}, \end{cases}$$

where $a^2 = ((p-1)^2 - 2)/(2(p-1))$, $b^2 = (p-1)/2$, $c^2 = 1/(2(p-1))$, $c > 0$ (here we use $p \ge 3$). The vector $\mathbf{z}_1$ has one entry $-a$ and $p + 1$

entries equal to c. Each of the vectors $\mathbf{z}_2, \dots, \mathbf{z}_n$ has one entry $-b$ and $p-1$ entries c. So

$$\|\mathbf{z}_1\|^2 = a^2 + (p+1)c^2 = \tfrac{1}{2}p,$$

$$\|\mathbf{z}_2\|^2 = \cdots = \|\mathbf{z}_n\|^2 = b^2 + (p-1)c^2 = \tfrac{1}{2}p.$$

If v_i and v_j are not adjacent, the corresponding vectors do not have common non-zero entries, and $\|\mathbf{z}_i - \mathbf{z}_j\|^2 = \|\mathbf{z}_i\|^2 + \|\mathbf{z}_j\|^2 = p$. If they are adjacent, the vectors have one common non-zero entry, and $\|\mathbf{z}_i - \mathbf{z}_j\|^2 = \|\mathbf{z}_i\|^2 + \|\mathbf{z}_j\|^2 + 2bc = p + 1$. We note that $[z_{ij}]$ is an upper triangular matrix with non-zero diagonal entries z_{ii}. Since the matrix is of full rank, the hyperplane spanned by $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$ does not contain the origin. As the square distance from the origin to any $\mathbf{z}_i$ is $\frac{1}{2}p$, the center of the sphere circumscribed to $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$ is the orthogonal projection of the origin onto the hyperplane spanned by $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$, and the square radius of the sphere is strictly less than $\frac{1}{2}p$. ▮

Proof of Theorem 2.4. It was proven in [4] that for any k and p there exists a regular graph H of degree p with girth greater than k. Let H' be a graph consisting of H_1, H_2, H_3, three vertex-disjoint copies of H, and let $v_i \in V(H_i)$, $i = 1,2,3$ be three vertices. Let $v_4 \in V(H_3)$ be a vertex adjacent to v_3. Now we omit the edge $\{v_3, v_4\}$ and add the edges $\{v_1, v_3\}$ and $\{v_2, v_4\}$. The resulting graph satisfies all the requirements. ▮

For a multigraph G, we denote by $v(G)$ and $e(G)$, respectively, the number of its vertices and the number of edges (which are counted with their multiplicities). A multigraph is called *dense* if for any induced subgraph $G' \neq G$, the strict inequality $\lambda(G') < \lambda(G)$ holds. As $\lambda(G') \leq \lambda(G)$ for any subgraph G' of G, one can always find a subgraph G' which is dense and satisfies $\lambda(G') = \lambda(G)$; we name G' a *maximal dense subgraph* (we do not claim that one is unique). If G is dense, its maximal dense subgraph is itself. The following result will be used in the proof of Theorem 2.5.

LEMMA 3.1 ([1, Lemma 1]). *If a multigraph G is dense, any two of its vertices are joined by at least one edge.*

Let G be a multigraph with vertices $v_1, v_2, \dots, v_n$ and adjacency matrix $[a_{ij}]$. We denote by $G(N_1, N_2, \dots, N_n)$ a *blow up* of G that is a multigraph whose vertices are divided into groups $V_1, V_2, \dots, V_n$ of size $N_1, N_2, \dots, N_n$, respectively, where any $u \in V_i$ and $w \in V_j$ are joined by a_{ij} edges. Obviously, the number of edges of the blow up is $\frac{1}{2}\sum_{i,j=1}^n a_{ij} N_i N_j$.

Proof of Theorem 2.5. As α is a jump, there is $\beta > \alpha$ such that $\lambda(G) > \alpha$ implies $\lambda(G) \geq \beta$. Choose $k > (\beta + \alpha)/(\beta - \alpha)$. Let $\mathscr{F}$ be the family of all q-multigraphs on k vertices and at least $((\beta + \alpha)/2)\binom{k}{2}$ edges. Then for any $F \in \mathscr{F}$:

$$\lambda(F) \geq \frac{2e(F)}{k^2} \geq \frac{(\beta + \alpha)\binom{k}{2}}{k^2}$$
$$= \frac{\beta + \alpha}{2}\left(1 - \frac{1}{k}\right) > \frac{\beta + \alpha}{2}\left(1 - \frac{\beta - \alpha}{\beta + \alpha}\right) = \alpha.$$

We modify $\mathscr{F}$ to a family $\mathscr{F}'$ by replacing every $F \in \mathscr{F}$ with its maximal dense subgraph. Clearly, $\mathscr{F}'$ is finite, and $\lambda(F) > \alpha$ for every $F \in \mathscr{F}'$. All multigraphs from $\mathscr{F}'$ are dense, and every multigraph with k vertices and more than $((\beta + \alpha)/2)\binom{k}{2}$ edges contains an $F \in \mathscr{F}'$ as a subgraph. We claim that $\{H: \lambda(H) \leq \alpha\} = \mathrm{Forb}(\mathscr{F}')$. Since $\lambda(F) > \alpha$ for every $F \in \mathscr{F}'$, it is sufficient to check that any H with $\lambda(H) > \alpha$ contains an $F \in \mathscr{F}'$. Indeed, if $\lambda(H) > \alpha$ then $\lambda(H) \geq \beta$. Let $V(H) = \{v_1, v_2, \dots, v_n\}$. Denote by a_{ij} the number of edges which join v_i and v_j. By continuity, there exist rational $x_1 = N_1/N$, $x_2 = N_2/N, \dots, x_n = N_n/N$ such that $x_1 + x_2 + \cdots + x_n = 1$ and $\sum_{i,j=1}^{n} a_{ij}x_ix_j > (\beta + \alpha)/2$. Then for the blow up $H' = H(N_1, N_2, \dots, N_n)$, we have

$$e(H') = \tfrac{1}{2}N^2 \sum_{i,j=1}^{n} a_{ij}x_ix_j > \frac{\beta + \alpha}{2}\binom{v(H')}{2}.$$

By simple averaging, one may find in H' an induced subgraph H'' with k vertices such that

$$\frac{e(H'')}{\binom{k}{2}} \geq \frac{e(H')}{\binom{v(H')}{2}}.$$

As H'' has more than $((\beta + \alpha)/2)\binom{k}{2}$ edges, it contains an $F \in \mathscr{F}'$ as a subgraph. Since F is dense, any two of its vertices are joined by at least one edge (by Lemma 3.1). Thus all the vertices of F must belong to different parts of the blow up $H' = H(N_1, N_2, \dots, N_n)$. So we conclude that F is contained in H as well.

Now we are going to show that the existence of finite $\mathscr{F}$ with $\mathrm{Forb}(\mathscr{F}) = \{H: \lambda(H) \leq \alpha\}$ implies that α is a jump. Indeed, if $\lambda(H) > \alpha$ then H

contains an $F \in \mathscr{F}$, and consequently $\lambda(H) \geq \beta$, where

$$\beta = \min_{F \in \mathscr{F}} \lambda(F) > \alpha. \quad \blacksquare$$

Proof of Lemma 2.6. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $I_k = \{i \in \{1, 2, \ldots, n\}: v_i \in V(G_k)\}$ for $k = 0, 1, \ldots, l$. Denote by a_{ij} the number of edges which join v_i and v_j in G. Consider non-negative $x_1, x_2, \ldots, x_n$ such that $x_1 + x_2 + \cdots + x_n = 1$. Set

$$X_k = \sum_{i \in I_k} x_i.$$

We note that

$$\sum_{i \in I_k} \sum_{j \in I_k} a_{ij} \frac{x_i}{X_k} \frac{x_j}{X_k} \leq \lambda(G_k).$$

We also note that $a_{ij} = t$ whenever $i \in I_k$, $j \in I_m$, $k \neq m$. Thus

$$\begin{aligned}
\sum_{i,j=1}^{n} a_{ij} x_i x_j &= t \sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j + \sum_{i,j=1}^{n} (a_{ij} - t) x_i x_j \\
&= t + \sum_{k=0}^{l} \sum_{i \in I_k} \sum_{j \in I_k} (a_{ij} - t) x_i x_j \\
&= t + \sum_{k=0}^{l} \left(-t(X_k)^2 + \sum_{i \in I_k} \sum_{j \in I_k} a_{ij} x_i x_j \right) \\
&\leq t + \sum_{k=0}^{l} (X_k)^2 (\lambda(G_k) - t) \\
&= t - \sum_{k=0}^{l} \frac{1}{\varepsilon_k} (X_k)^2.
\end{aligned}$$

The maximum of the last expression is attained when $X_k = \varepsilon_k / (\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_l)$ for $k = 0, 1, \ldots, l$. Therefore,

$$\lambda(G) \leq t - \sum_{k=0}^{l} \frac{\varepsilon_k}{(\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_l)^2} = t - \frac{1}{\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_l}.$$

Now let $\{y_i | i \in I_k\}$ be the reals for which the maximum of the quadratic form of G_k is attained:

$$\sum_{i\in I_k}\sum_{j\in I_k} a_{ij}y_iy_j = \lambda(G_k), \qquad \sum_{i\in I_k} y_i = 1,\ y_i \geq 0 \text{ for } i \in I_k.$$

If we set $x_i = (\varepsilon_k/(\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_l))y_i$ for every $i \in I_k$ and every $k = 0, 1, \ldots, l$, all the inequalities become equalities. ∎

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